Homotopy BV-algebra structure on the double cobar construction

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Abstract

We show that the double cobar construction, $\Omega^2 C_*(X)$, of a simplicial set X is a homotopy BV-algebra if X is a double suspension, or if X is 2-reduced and the coefficient ring contains the field of rational numbers $\mathbb{Q}$. Indeed, the Connes-Moscovici operator defines the desired homotopy BV-algebra structure on $\Omega^2 C_*(X)$ when the antipode $S : \Omega C_*(X) \to \Omega C_*(X)$ is involutive. We proceed by defining a family of obstructions $O_n : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes n}$, $n \geq 2$ by computing $S^2 - Id$. When X is a suspension, the only obstruction remaining is $O_2 := E^{1,1} - \tau E^{1,1}$ where $E^{1,1}$ is the dual of the $\smile_1$-product. When X is a double suspension the obstructions vanish.

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Introduction

Adams' cobar construction provides a model of the loop space of a 1-connected topological space [1]. The cobar construction is a functor from differential graded coalgebras to differential graded algebras; for iteration, a coproduct is thus needed. For a 1-reduced simplicial set X, Baues defined [4] a DG-bialgebra structure on its first cobar construction $\Omega C_*(X)$. The resulting double cobar construction is an algebraic model for the double loop space.
We show that the double cobar construction, $\Omega^2 C_*(X)$, of a simplicial set X is a homotopy BV-algebra (a homotopy G-algebra in the sense of Gerstenhaber-Voronov [7] together with a degree one operator) if X is a double suspension, or if X is 2-reduced and the coefficient ring contains the field of rational numbers $\mathbb{Q}$.

Baues' coproduct on $\Omega C_*(X)$ is equivalent to a homotopy G-coalgebra structure $\{E^{k,1}\}_{k\geq 1}$ on the DG-coalgebra $C_*(X)$, that is a family of operations

$$E^{k,1} : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes k} \otimes \widetilde{C}_*(X), \;\; k \geq 1,$$

satisfying some relations. This corresponds also to the coalgebra structure of $C_*(X)$ over the second stage filtration operad $F_2\chi$ of the surjection operad χ given in [5, 16], see Section 4.1. Since a bialgebra structure determines the antipode (whenever it exists), the antipode

$$S : \Omega C_*(X) \to \Omega C_*(X)$$

on the cobar construction is then determined by the homotopy G-coalgebra structure on $C_*(X)$.

As a model for the chain complex of double loop spaces, the double cobar construction is expected to be a Batalin-Vilkovisky algebra up to homotopy. Indeed, it is well-known that the circle action on the double loop space $\Omega^2 X$ by rotating the equator defines a BV operator; thereby the homology $H_*(\Omega^2 X)$ is a Batalin-Vilkovisky algebra [8].

The cobar construction $\Omega\mathcal{H}$ of an involutive Hopf algebra turns out to be the underlying complex in the Hopf-cyclic Hochschild cohomology of $\mathcal{H}$, [6]. The cyclic operator requires an involutive antipode on the underlying bialgebra $\mathcal{H}$. Assuming that the antipode is involutive, Menichi proved [17], that for a unital (ungraded) Hopf algebra $\mathcal{H}$, the Connes-Moscovici operator induces a Batalin-Vilkovisky algebra on the homology of the cobar construction $H_*(\Omega\mathcal{H})$.

By defining a family of operations

$$O_n : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes n}, \;\; n \geq 2,$$

we provide a criterion for the involutivity of S in terms of the operations

$$E^{k,1} : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes k} \otimes \widetilde{C}_*(X).$$

By Kadeishvili's work [11] the double cobar construction is a homotopy G-algebra, it is endowed with a family of operations

$$E_{1,k} : \Omega^2 C_*(X) \otimes \left(\Omega^2 C_*(X)\right)^{\otimes k} \to \Omega^2 C_*(X), \;\; k \geq 1,$$

satisfying some relations. In particular, the following bracket

$$\{a; b\} = E_{1,1}(a \otimes b) - (-1)^{(|a|+1)(|b|+1)} E_{1,1}(b \otimes a), \;\; a, b \in \Omega^2 C_*(X),$$

together with the DG-product of $\Omega^2 C_*(X)$, induces a Gerstenhaber algebra structure on the homology $H_*(\Omega^2 C_*(X))$. The vanishing of the operations

$$O_n : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes n}, \;\; n \geq 2,$$

gives a sufficient condition for the extension of the homotopy G-algebra structure on $\Omega^2 C_*(X)$ given by Kadeishvili to the homotopy BV-algebra structure whose BV-operator is the Connes-Moscovici operator. We establish the following for a general homotopy G-coalgebra,

Proposition 3.3. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra.*

1. *The cobar construction ΩC is an involutive DG-Hopf algebra if and only if all the obstructions $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$ defined in (3.5) for $n \geq 2$ are zero.*
2. *Let C be 2-reduced i.e. $C_0 = R$ and $C_1 = C_2 = 0$. If all the obstructions $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra given by the Connes-Moscovici operator.*

We apply this criterion to the homotopy G-coalgebra $C_*(X)$ of a 1-reduced simplicial set X. We show that, when ΣX is a simplicial suspension, the family of operations

$$O_n : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X)^{\otimes n}$$

reduces to

$$O_2 : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X) \otimes \widetilde{C}_*(\Sigma X).$$

This operation O_2 is the deviation from the cocommutativity of the operation

$$E^{1,1} : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X) \otimes \widetilde{C}_*(\Sigma X).$$

In fact, $E^{1,1}$ is the only non-trivial operation of the family $\{E^{k,1}\}_{k\geq 1}$ defining the homotopy G-algebra structure on $C_*(\Sigma X)$, see Propositions 4.4 and 4.5. However, in the case of a double simplicial suspension $\Sigma^2 X$, the operation $E^{1,1}$ is also trivial, and all the obstructions

$$O_n : \widetilde{C}_*(\Sigma^2 X) \to \widetilde{C}_*(\Sigma^2 X)^{\otimes n}, \;\; n \geq 2,$$

are zero. As a consequence, the cobar construction $\Omega C_*(\Sigma^2 X)$ is the free tensor DG-algebra with the shuffle coproduct. Thus we have,

Theorem 4.6. *Let $\Sigma^2 X$ be a double suspension. Then:*

- *the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ corresponding to Baues' coproduct*

  $$\nabla_0 : \Omega C_*(\Sigma^2 X) \to \Omega C_*(\Sigma^2 X) \otimes \Omega C_*(\Sigma^2 X),$$

  has trivial higher operations i.e. $E^{1,k} = 0$ for $k \geq 1$;
- *the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is a homotopy BV-algebra with the Connes-Moscovici operator as BV-operator.*

On the other hand, if the ground ring contains the field of rational numbers $\mathbb{Q}$, we deform (see Section 4.4 for the precise statement) $(\Omega C_*(X), \nabla_0, S)$ into a cocommutative DG-Hopf algebra $(\Omega C_*(X), \nabla_0', S')$. Therefore, $(\Omega C_*(X), \nabla_0', S')$ has an involutive antipode and the homotopy BV-algebra structure we consider on the deformed double cobar construction $\Omega(\Omega C_*(X), \nabla_0', S')$ follows. Thus, we have

Theorem 4.8. *Let X be a 2-reduced simplicial set. Then the double cobar construction $\Omega(\Omega C_*(X), \nabla_0', S')$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator the Connes-Moscovici operator.*

The paper is organized as follows.

In the first section we review background materials on the bar-cobar constructions and Hopf algebras.

The second section is devoted to the structures of homotopy G-algebras [11] and homotopy BV-algebras [17] on the cobar construction. These first two sections fix notations and sign conventions.

In the third section we define the family of operations $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$, $n \geq 2$, on a homotopy G-coalgebra $(C, E^{k,1})$.

In the section 4 we give applications :
In the subsection 4.1 we set the convention for the homotopy G-coalgebra structure on $C_*(X)$. We compare the homotopy G-coalgebra structure on $C_*(X)$ coming from Baues' coproduct with the action of the surjection operad given in [5, 16].
Subsections 4.2 and 4.3 give applications to simplicial suspensions. In the case of single suspension ΣX the family of operations $O_n : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X)^{\otimes n}$, $n \geq 2$ reduces to O_2. We show that, for a double suspension, this last obstruction O_2 vanishes.
The last subsection 4.4 is devoted to the rational case. We prove that the double cobar construction of a 2-reduced simplicial set is a homotopy BV-algebra.
In Appendix we recall and specify some facts about the Hirsch and the homotopy G-algebras. In particular, we make explicit the signs related to our sign convention.

1. Notations and preliminaries

1.1. *Conventions and notations*

Let R be a commutative ring. A graded R-module M is a family of R-modules $\{M_n\}$ where indices n run through the integers. The degree of $a \in M_n$ is denoted by $|a|$, so here $|a| = n$. The r-suspension s^r is defined by $(s^r M)_n = M_{n-r}$.
Algebras (respectively coalgebras) are understood as associative algebras (respectively coassociative coalgebras).
A unital R-algebra (A, μ, η) is called augmented if there is an algebra morphism $\epsilon : A \to R$. We denote by $\overline{A}$ the augmentation ideal $\mathrm{Ker}(\epsilon)$.
For a coalgebra (C, ∇) the n-iterated coproduct is denoted by

$$\nabla^{(n)} : C \to C^{\otimes n+1}$$

for $n \geq 1$. We use the Sweedler notation,

$$\nabla(c) = c^1 \otimes c^2 \quad \text{and} \quad \nabla^{(n)}(c) = c^1 \otimes c^2 \otimes \cdots \otimes c^{n+1},$$

where we have omitted the sum. A counital R-coalgebra (C, ϵ) is called coaugmented if there is a coalgebra morphism $\eta : R \to C$. We denote by $\overline{C} = \mathrm{Ker}(\epsilon)$ the reduced coalgebra with the reduced coproduct

$$\overline{\nabla}(c) = \nabla(c) - c \otimes 1 - 1 \otimes c.$$

A (co)algebra A is called connected if it is both (co)augmented and $A_n = 0$ for $n \leq -1$ and $A_0 \cong R$.
A (co)algebra, A is called n-reduced if it is both connected and $A_k = 0$ for $1 \leq k \leq n$.
A coaugmented coalgebra C is called conilpotent if the following filtration

$$\begin{aligned} &F_0 C := R \\ &F_r C := R \oplus \{c \in \overline{C} | \nabla^n(c) = 0, n \geq r\} \ \text{ for } \ r \geq 1, \end{aligned}$$

is exhaustive, that is $C = \bigcup_r F_r C$.

1.2. *The bar and cobar constructions*

We refer to [14, Chapter 2] for the background materials related to the bar and cobar constructions.

The cobar construction is a functor

$$\begin{aligned} \Omega : DGC_1 &\to DGA_0 \\ (C, d_C, \nabla_C) &\mapsto \Omega C = (T(s^{-1}\,\overline{C}), d_\Omega) \end{aligned}$$

from the category of 1-connected DG-coalgebras to the category of connected DG-algebras. Here, $T(\mathrm{s}^{-1}\,\overline{C})$ is the free tensor algebra on the module $\mathrm{s}^{-1}\,\overline{C}$ and d_Ω is the unique derivation such that $d_\Omega(\mathrm{s}^{-1}\,c) = -\,\mathrm{s}^{-1}\,d_{\overline{C}}(c) + (\mathrm{s}^{-1} \otimes \mathrm{s}^{-1})\overline{\nabla}_C(c)$ for all $c \in \overline{C}$.

The bar construction is a functor

$$\begin{aligned} \mathcal{B} : DGA_0 &\to DGC_c \\ (A, d_A, \mu_A) &\mapsto \mathcal{B}\,A = (T^c(\mathrm{s}\,\overline{A}), d_{\mathcal{B}}) \end{aligned}$$

from the category of connected DG-algebras to the category of conilpotent DG-coalgebras. Here, $T^c(\mathrm{s}\,\overline{A})$ is the cofree tensor coalgebra on the module $\mathrm{s}\,\overline{A}$ and $d_{\mathcal{B}}$ is the unique coderivation with components

$$T^c(\mathrm{s}\,\overline{A}) \twoheadrightarrow \mathrm{s}\,\overline{A} \oplus (\mathrm{s}\,\overline{A})^{\otimes 2} \xrightarrow{-\,\mathrm{s}\,d_{\overline{A}}\mathrm{s}^{-1} + \mathrm{s}\,\mu_{\overline{A}}(\mathrm{s}^{-1}\otimes \mathrm{s}^{-1})} \mathrm{s}\,\overline{A}.$$

We recall the bar-cobar adjunction,

Theorem 1.1. *[14, Theorem 2.2.9] For every augmented DG-algebra Λ and every conilpotent DG-coalgebra C there exist natural bijections*

$$\mathrm{Hom}_{\mathit{DG\text{-}Alg}}(\Omega C, \Lambda) \cong \mathrm{Tw}(C; \Lambda) \cong \mathrm{Hom}_{\mathit{DG\text{-}Coalg}}(C, \mathcal{B}\,\Lambda).$$

The set $\mathrm{Tw}(C;\Lambda)$ of twisting cochains from C to Λ is the set of degree 1 linear maps $f : C \to \Lambda$ verifying the twisting condition: $\partial f := df + fd = -\mu_\Lambda(f \otimes f)\nabla_C$.

1.3. Hopf algebras

Definition 1.2. For a DG-bialgebra $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon)$ an **antipode** $S : \mathcal{H} \to \mathcal{H}$ is a chain map which is the inverse of the identity in the convolution algebra $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$; the convolution product being $f \smile g = \mu(f \otimes g)\nabla$. Explicitly, S satisfies

$$S(a^1)a^2 = \eta\epsilon(a) = a^1 S(a^2)$$

for all $a \in \mathcal{H}$.

Definition 1.3. A **DG-Hopf algebra** $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon, S)$ is a DG-bialgebra $\mathcal{H}$ endowed with an antipode S. If moreover, the antipode is involutive (i.e. $S^2 = id$) we call $\mathcal{H}$ an **involutive DG-Hopf algebra**.

An antipode satisfies the following properties.

Proposition 1.4. *[19, Proposition 4.0.1]*

i. $S(a^2) \otimes S(a^1) = (-1)^{|a^1||a^2|} S(a)^1 \otimes S(a)^2$ *(coalgebra antimorphism).*
ii. $S(\eta(1)) = \eta(1)$ *(unital morphism).*
iii. $S(ab) = (-1)^{|a||b|} S(b)S(a)$ *(algebra antimorphism).*
iv. $\epsilon(S(a)) = \epsilon(a)$ *(counital morphism).*
v. *The following equations are equivalent:*
   (a) $S^2 = S \circ S = id$;
   (b) $S(a^2)a^1 = \eta\epsilon(a)$;
   (c) $a^2 S(a^1) = \eta\epsilon(a)$.
vi. *If $\mathcal{H}$ is commutative or cocommutative, then $S^2 = id$.*

2. Homotopy structures on the cobar construction

2.1. Homotopy G-algebra on the cobar construction

We present the homotopy G-algebra structure on the cobar construction of a 1-reduced DG-bialgebra given in [11]. The homotopy G-algebras are also known as the Gerstenhaber-Voronov algebras defined in [7]. We define the Hirsch algebras and the homotopy G-algebras via the bar construction.

Definition 2.1. A **Hirsch algebra** Λ is the data of a connected DG-algebra $(\Lambda, d, \mu_\Lambda)$ together with a map $\mu : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \to \mathcal{B}\Lambda$ making $\mathcal{B}\Lambda$ into an associative unital DG-bialgebra.

For a connected DG-algebra Λ a DG-product $\mu : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \to \mathcal{B}\Lambda$ corresponds to a twisting cochain $\widetilde{E} \in \mathrm{Tw}(\mathcal{B}\Lambda \otimes \mathcal{B}\Lambda, \Lambda)$, cf. Theorem 1.1. After a correct use of desuspensions (see sign convention below), the latter is a family of operations $\{E_{i,j}\}_{i,j\geq 1}$, $E_{i,j} : \overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j} \to \overline{\Lambda}$ satisfying some relations, see [11]. We denote by $\mu_E : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \to \mathcal{B}\Lambda$ such a DG-product.

A Hirsch algebra is a particular B_∞-algebra whose underlying A_∞-algebra structure is a DGA structure. Recall that the bar construction $\mathcal{B}\Lambda$ of a DG-algebra Λ is the cofree tensor coalgebra $T^c(s\overline{\Lambda})$ together with the derivation induced by the differential and the product of Λ.

	Λ		$T^c(s\overline{\Lambda})$
DGA	(d_Λ, μ_Λ)	DGC	d induced by d_Λ and μ_Λ
A_∞	$\{\mu_k\}_{k\geq 0}$	DGC	d induced by μ_k for $k \geq 0$
Hirsch	$(d_\Lambda, \mu, \{E_{i,j}\}_{i,j\geq 1})$	DG-bialgebra	d induced by d_Λ and μ ; DG-product μ_E
B_∞	$(\{\mu_k\}_{k\geq 0}, \{E_{i,j}\}_{i,j\geq 1})$	DG-bialgebra	d induced by μ_k for $k \geq 0$; DG-product μ_E

Definition 2.2. A **homotopy G-algebra** $(\Lambda, d_\Lambda, \mu_\Lambda, \{E_{1,j}\}_{j\geq 1})$ is a Hirsch algebra $(\Lambda, d_\Lambda, \mu_\Lambda, \{E_{i,j}\}_{i,j\geq 1})$ such that $E_{i,j} = 0$ for $i \geq 2$.

Sign convention

Let $\widetilde{E} \in \mathrm{Tw}(\mathcal{B}\Lambda \otimes \mathcal{B}\Lambda, \Lambda)$ be a twisting cochain. Its (i,j)-component is $\widetilde{E}_{i,j} : (\mathrm{s}\overline{\Lambda})^{\otimes i} \otimes (\mathrm{s}\overline{\Lambda})^{\otimes j} \to \overline{\Lambda}$. We denote by $E_{i,j} : \overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j} \to \overline{\Lambda}$ the component

$$\begin{aligned} &E_{i,j}(a_1 \otimes \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_{i+j}) \\ &\qquad := (-1)^{\sum_{s=1}^{i+j-1} |a_s|(i+j-s)} \widetilde{E}_{i,j}(\mathrm{s}^{\otimes i} \otimes \mathrm{s}^{\otimes j})(a_1 \otimes \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_{i+j}) \\ &\qquad = \widetilde{E}_{i,j}(\mathrm{s}\, a_1 \otimes \ldots \otimes \mathrm{s}\, a_i \otimes \mathrm{s}\, a_{i+1} \otimes \ldots \otimes \mathrm{s}\, a_{i+j}). \end{aligned} \tag{2.1}$$

The degree of $E_{i,j}$ is $i + j - 1$. We denote $E_{i,j}((a_1 \otimes \ldots \otimes a_i) \otimes (b_1 \otimes \ldots \otimes b_j))$ by $E_{i,j}(a_1, \ldots, a_i; b_1, \ldots b_j)$.

Let B be a 1-reduced DG-bialgebra. For $a \in B$ and $\overline{b} := \mathrm{s}^{-1} b_1 \otimes \ldots \otimes \mathrm{s}^{-1} b_s \in \Omega B$, we set

$$a \diamond \overline{b} = \mathrm{s}^{-1}(a^1 b_1) \otimes \ldots \otimes \mathrm{s}^{-1}(a^s b_s).$$

Proposition 2.3. *[11] Let B be a 1-reduced DG-bialgebra. Then the cobar construction $(\Omega B, d_\Omega)$ together with the operations $E_{1,k}$, $k \geq 1$ defined by*

$$\begin{aligned} &E_{1,k}(\mathrm{s}^{-1} a_1 \otimes \ldots \otimes \mathrm{s}^{-1} a_n; \overline{b}_1, \ldots, \overline{b}_k) = \\ &\sum_{1\leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n} \pm\, \mathrm{s}^{-1} a_1 \otimes \ldots \otimes \mathrm{s}^{-1} a_{i_1 - 1} \otimes a_{i_1} \diamond \overline{b}_1 \otimes \ldots \otimes \mathrm{s}^{-1} a_{i_k - 1} \otimes a_{i_k} \diamond \overline{b}_k \otimes \mathrm{s}^{-1} a_{i_k + 1} \otimes \ldots \otimes \mathrm{s}^{-1} a_n \end{aligned} \tag{2.2}$$

when $n \geq k$ and zero when $n < k$ is a homotopy G-algebra.

In particular the operation $E_{1,1} : \overline{\Omega B} \otimes \overline{\Omega B} \to \Omega B$ is given by

$$E_{1,1}(\mathrm{s}^{-1} a_1 \otimes ... \otimes \mathrm{s}^{-1} a_m; \mathrm{s}^{-1} b_1 \otimes ... \otimes \mathrm{s}^{-1} b_n)$$
$$:= \sum_{l=1}^{m} (-1)^{\gamma} \mathrm{s}^{-1} a_1 \otimes ...(a_l^1 \cdot b_1) \otimes ... \otimes \mathrm{s}^{-1}(a_l^n \cdot b_n) \otimes ... \otimes \mathrm{s}^{-1} a_m \quad (2.3)$$

with

$$\gamma = \sum_{u=1}^{n} |b_u| \left(\sum_{s=l+u}^{m} |a_s| + m - l - u + 1 \right) + \kappa_n(a_l) + \sum_{u=2}^{n} (|a_l^u| - 1) \left(\sum_{s=1}^{u-1} |b_s| - u + 1 \right)$$
$$+ \sum_{s=1}^{n} (|a_l^s| - 1)(2n - 2s + 1) + \sum_{s=1}^{n-1} (|a_l^s| + |b_s|)(n - s),$$

where

$$\kappa_n(a) := \begin{cases} \sum_{1 \le 2s+1 \le n} |a^{2s+1}| & \text{if } n \text{ is even;} \\ \sum_{1 \le 2s \le n} |a^{2s}| & \text{if } n \text{ is odd.} \end{cases}$$

We postpone the construction of this operation to Appendix.

2.2. Homotopy BV-algebra on the cobar construction

We define a homotopy BV-algebra as a homotopy G-algebra (in the sense of Gerstenhaber-Voronov [7]) together with a BV-operator. The main example, that we will discuss in details, is the cobar construction of a 1-reduced involutive DG-Hopf algebra $\mathcal{H}$. The BV-operator on $\Omega\mathcal{H}$ is the Connes-Moscovici operator defined in [6].

Definition 2.4. A **homotopy BV-algebra** Λ is a homotopy G-algebra together with a degree one DG-operator $\Delta : \Lambda \to \Lambda$ subject to the relations:

$$\Delta^2 = 0;$$
$$\{a; b\} = (-1)^{|a|} \big(\Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|} a \cdot \Delta(b) \big)$$
$$+ d_\Lambda H(a; b) + H(d_\Lambda a; b) + (-1)^{|a|} H(a; d_\Lambda b) \text{ for all } a, b \in \Lambda,$$

where $\{-;-\}$ is the Gerstenhaber bracket $\{a; b\} = E_{1,1}(a; b) - (-1)^{(|a|+1)(|b|+1)} E_{1,1}(b; a)$ and $H : \Lambda \otimes \Lambda \to \Lambda$ is a degree 2 linear map.

A straightforward calculation shows that:

$$\Delta(\{a; b\}) = \{\Delta(a); b\} + (-1)^{|a|+1} \{a; \Delta(b)\} - \partial \overline{H}(a; b),$$

where

$$\overline{H}(a; b) := \Delta H(a; b) + H(\Delta(a); b) + (-1)^{|a|} H(a; \Delta(b)) \;\; \forall a, b \in \Lambda.$$

Therefore Δ is a derivation for the bracket $\{-;-\}$ if Δ is a derivation for H, or more generally if Δ is a derivation for H up to a homotopy.
In the example we consider hereafter, the homotopy H is itself antisymmetric, more precisely, we have $H(a; b) := H_1(a; b) - (-1)^{(|a|+1)(|b|+1)} H_1(b; a)$. It turns out that the operator Δ is not a derivation for H in general. However we do not know yet if there exists a homotopy H' such that $\overline{H} = \partial H'$.

A. Connes and H. Moscovici defined in [6] a boundary map on the cobar construction of an involutive Hopf algebra $\mathcal{H}$. This operator, hereafter called the Connes-Moscovici operator, induces a BV-operator on the homology $H_*(\Omega\mathcal{H})$. More precisely, let $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon, S)$ be an involutive DG-Hopf algebra, that is with an involutive antipode S. With our sign convention, the Connes-Moscovici operator

$$\Delta : \Omega\mathcal{H} \to \Omega\mathcal{H}$$

is zero on both $R = (s^{-1}\overline{\mathcal{H}})^{\otimes 0}$ and $(s^{-1}\overline{\mathcal{H}})^{\otimes 1}$, and is given on the n-th component $(s^{-1}\overline{\mathcal{H}})^{\otimes n}$, $n \geq 2$ by:

$$\Delta(s^{-1} a_1 \otimes s^{-1} a_2 \otimes ... \otimes s^{-1} a_n) = \sum_{i=0}^{n-1} (-1)^i \pi_n \tau_n^i (s^{-1} a_1 \otimes s^{-1} a_2 \otimes ... \otimes s^{-1} a_n), \tag{2.4}$$

where τ_n is the cyclic permutation

$$\tau_n(s^{-1} a_1 \otimes s^{-1} a_2 \otimes ... \otimes s^{-1} a_n) := (-1)^{(|a_1|-1)(\sum_{i=2}^n |a_i|-1)} s^{-1} a_2 \otimes s^{-1} a_3 \otimes ... \otimes s^{-1} a_n \otimes s^{-1} a_1,$$

and

$$\pi_n := (s^{-1}\mu)^{\otimes n-1} \tau_{n-1,n-1} (S^{\otimes n-1} \otimes 1^{\otimes n-1})((\tau\nabla)^{(n-2)} \otimes 1^{n-1})\, s^{\otimes n},$$

$$\tau_{n,n}(a_1 \otimes a_2 \otimes ... \otimes a_n \otimes b_1 \otimes ... \otimes b_n) := (-1)^{\sum_{i=1}^{n-1} |b_i| (\sum_{j=i+1}^n |a_j|)} a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes ... \otimes a_n \otimes b_n.$$

More explicitly,

$$\Delta(s^{-1} a_1 \otimes s^{-1} a_2 \otimes ... \otimes s^{-1} a_n) = \sum_{k=1}^n \pm s^{-1} S(a_k^{n-1}) a_{k+1} \otimes s^{-1} S(a_k^{n-2}) a_{k+2} \otimes ... \otimes s^{-1} S(a_k^1) a_{k-1}. \tag{2.5}$$

The involutivity of the antipode of $\mathcal{H}$ makes Δ into a square zero chain map. L. Menichi proved [17, Proposition 1.9] that for a unital (ungraded) Hopf algebra $\mathcal{H}$, the Connes-Moscovici operator induces a Batalin-Vilkovisky algebra on the homology of the cobar construction $H_*(\Omega\mathcal{H})$. In our context,

Proposition 2.5. *[17] Let $\mathcal{H}$ be a 1-reduced involutive DG-Hopf algebra. Then the cobar construction $(\Omega\mathcal{H}, d_\Omega)$ is a homotopy BV-algebra whose the BV-operator is defined in (2.4).*

Proof. Let us define Menichi's homotopy $H(\overline{a};\overline{b}) := H_1(\overline{a};\overline{b}) - (-1)^{(|\overline{a}|+1)(|\overline{b}|+1)} H_1(\overline{b};\overline{a})$.

$$\begin{aligned} &H_1(s^{-1} a_1 \otimes ... \otimes s^{-1} a_m; s^{-1} b_1 \otimes ... \otimes s^{-1} b_n) \\ &:= \sum_{1 \leq j \leq p \leq m-1} (-1)^{\xi_j + n + 1} \pi^s_{m+n-1} \tau^{s, n+m-1-j}_{m+n-1} \rho^{(p-j+1)}_{n+m} (s^{-1} a_1 \otimes ... \otimes s^{-1} a_m \otimes s^{-1} b_1 \otimes ... \otimes s^{-1} b_n) \end{aligned} \tag{2.6}$$

with

$$\xi_j = \begin{cases} \displaystyle\sum_{s=1}^m |a_s| & \text{for } j = 1; \\ \displaystyle\sum_{s=m-j+1}^m |a_s| & \text{for } j > 1. \end{cases}$$

Where,

$$\begin{aligned} \pi^s_m &= \pi_m (s^{-1})^{\otimes m} \\ \tau^{s,i}_m &:= (\tau^s_m)^i \\ \tau_m(a_1 \otimes a_2 \otimes ... \otimes a_m) &:= (-1)^{|a_1|(\sum_{i=2}^n |a_i|)} a_2 \otimes a_3 \otimes ... \otimes a_m \otimes a_1 \\ \rho^{(i)}_{m+1} &:= (1^{\otimes i-1} \otimes \mu \otimes 1^{\otimes m-i})(1^{\otimes i-1} \otimes \tau_{m-i+1,1}) s^{\otimes m+1} \\ \tau_{k,1}(a_1 \otimes ... \otimes a_k \otimes b) &:= (-1)^{|b|(\sum_{i=2}^k |a_i|)} a_1 \otimes b \otimes a_2 \otimes ... \otimes a_k. \end{aligned}$$

By Proposition [17, Proposition 1.9] we have

$$\{a;b\} = (-1)^{|a|}\big(\Delta(a\cdot b) - \Delta(a)\cdot b - (-1)^{|a|}a\cdot\Delta(b)\big) + d_1 H(a;b) + H(d_1 a;b) + (-1)^{|a|}H(a;d_1 b)$$

on the cobar construction $(\Omega\mathcal{H}, d_0 + d_1)$, where d_1 is the quadratic part of the differential d_Ω. The operators involved in the above equation commute with d_0. Therefore we can replace d_1 by d_Ω in the previous equation. ■

3. Involutivity of the antipode of ΩC in terms of the homotopy G-coalgebra C

A Hirsch coalgebra C is the formal dual of a Hirsch algebra, that is it corresponds to a DG-coproduct $\nabla : \Omega C \to \Omega C \otimes \Omega C$ making ΩC into a DG-bialgebra. Baues' coproduct [4] $\nabla_0 : \Omega C_*(X) \to \Omega C_*(X)$ defined on the cobar construction of a simplicial set X corresponds to a homotopy G-coalgebra structure $E^{k,1}$ on $C_*(X)$, that is a particular case of Hirsch coalgebra structure. With this example in mind we consider a homotopy G-coalgebra $(C, E^{k,1})$. Then there exists a unique antipode $S : \Omega C \to \Omega C$ on its cobar construction. The purpose of this section is to give a criterion for the involutivity of the antipode S in terms of the operations $E^{k,1}$. This takes the form of a family of operations

$$O_n : \mathrm{s}^{-1}\,\overline{\mathrm{C}} \to (\mathrm{s}^{-1}\,\overline{\mathrm{C}})^{\otimes n}, \;\; n \geq 2.$$

For convenience, we set $\widetilde{C} := \mathrm{s}^{-1}\,\overline{\mathrm{C}}$. Thus $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$, $n \geq 2$.

Definition 3.1. A **Hirsch coalgebra** C is the data of a 1-reduced DG-coalgebra (C, d, ∇_C) together with a map $\nabla : \Omega C \to \Omega C \otimes \Omega C$ making ΩC into a coassociative counital DG-bialgebra. The corresponding operations on C are denoted by $E^{i,j} : \widetilde{C} \to \widetilde{C}^{\otimes i} \otimes \widetilde{C}^{\otimes j}$. The degree of $E^{i,j}$ is 0. A **homotopy G-coalgebra** is a Hirsch coalgebra whose operations $E^{i,j} = 0$ for $j \geq 2$.

Let the cobar construction $(\Omega C, d, \mu_\Omega, \nabla)$ be a DG-bialgebra. We can define the antipode

$$S : \Omega C \to \Omega C$$

by $S(\eta(1)) = \eta(1)$ and for $\sigma \in \widetilde{C}_n$ with $n \geq 1$ by

$$S(\sigma) := -\sigma - \mu_\Omega(S\otimes 1)\overline{\nabla}(\sigma) \tag{3.1}$$

which makes sense since

$$\overline{\nabla}(\sigma) \subset \bigoplus_{\substack{i+j=n\\ 0<i,j<n}} (\Omega C)_i \otimes (\Omega C)_j.$$

Indeed, $\mu_\Omega(S\otimes 1)\nabla(\eta(1)) = \eta\epsilon\eta(1) = \eta(1)$ gives immediately that $S(\eta(1)) = \eta(1)$. Moreover, since $\mu_\Omega(S\otimes 1)\nabla(\sigma) = 0$ for all $\sigma \in \widetilde{C}_n$ with $n \geq 1$, we have

$$\begin{aligned}\mu_\Omega(S\otimes 1)\nabla(\sigma) &= \mu_\Omega(S\otimes 1)\overline{\nabla}(\sigma) + \mu_\Omega(S(\sigma)\otimes\eta(1) + S(\eta(1))\otimes\sigma)\\ &= \mu_\Omega(S\otimes 1)\overline{\nabla}(\sigma) + S(\sigma) + \sigma = 0.\end{aligned}$$

Remark 3.2. Equivalently, we can define S, see [14, section 1.3.10 p.15], as the geometric serie $(Id)^{\smile -1} = \sum_{n\geq 0}(\eta\epsilon - Id)^{\smile n}$ with $(\eta\epsilon - Id)^{\smile 0} = \eta\epsilon$ and where $\smile$ denote the convolution product from Definition 1.2. Note that this sum is finite when it is evaluated on an element. This presentation is combinatorial and gives for $[\sigma] \in (\mathrm{s}^{-1}\,\overline{\mathrm{C}})_n$,

$$S([\sigma]) = \eta\epsilon([\sigma]) + (\eta\epsilon - Id)([\sigma]) + \mu_\Omega\overline{\nabla}([\sigma]) - \mu_\Omega^{(2)}\overline{\nabla}^{(2)}([\sigma]) + \ldots + (-1)^{n-1}\mu_\Omega^{(n)}\overline{\nabla}^{(n)}([\sigma]). \tag{3.2}$$

By the iii of Proposition 1.4 the antipode is an algebra antimorphism that is an algebra morphism from $(\Omega C, \mu_\Omega)$ to $\Omega C_{(12)} := (\Omega C, \mu_{\Omega_{(12)}} := \mu_\Omega \tau)$. Moreover, it is also a DG-map, therefore it corresponds to a twisting cochain $F \in \mathrm{Tw}(C, \Omega C_{(12)})$. An antipode is determined by the underlying bialgebra structure. Here the latter is equivalent to a homotopy G-algebra structure $E^{k,1}$ on C. We make explicit the twisting cochain F in terms of $E^{k,1}$.
We recall the notation $\widetilde{C} := s^{-1}\,\overline{C}$. We write $F^i : \widetilde{C} \to \widetilde{C}^{\otimes i}$ for the i-th component of F. The relation[1] $\mu_\Omega(1 \otimes S)\nabla = \eta\epsilon$ gives

$$F^1 = -Id_{\widetilde{C}}; \tag{3.3}$$

$$F^n = \sum_{\substack{1 \le s \le n-1 \\ n_1+\ldots+n_s=n-1 \\ n_i \ge 1}} (-1)^{s+1}(1^{\otimes n_1+\ldots+n_{s-1}} \otimes E^{n_s,1})\ldots(1^{\otimes n_1} \otimes E^{n_2,1})E^{n_1,1}, \quad n \ge 2. \tag{3.4}$$

Let the operation $E^{n_i,1} : \widetilde{C} \to \widetilde{C}^{\otimes n_i} \otimes \widetilde{C}$ be represented by the tree in Figure 1. Then the summands of the equation (3.4) are represented by the tree in Figure 2.

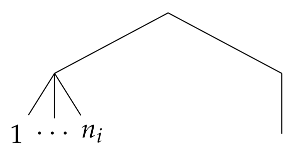

Figure 1: The operation $E^{n_i,1}$.

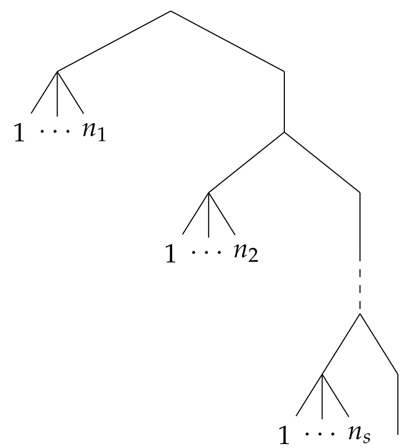

Figure 2: A summand of F^n

The three first terms are $F^1 = -Id_{\widetilde{C}}$, $F^2 = E^{1,1}$ and $F^3 = -(1 \otimes E^{1,1})E^{1,1} + E^{2,1}$.

Formulated in these terms, the n-th component of the difference $S^2 - Id$ is zero for $n = 1$ and

$$O_n := \sum_{s=1}^{n} \sum_{n_1+\ldots+n_s=n} \mu^{(s)}_{\Omega_{(12)}}(F^{n_1} \otimes \ldots \otimes F^{n_s})F^s \tag{3.5}$$

for $n \ge 2$. The terms $(F^{n_1} \otimes \ldots \otimes F^{n_s})F^s$ have codomain $\widetilde{C}^{\otimes n_1} \otimes \widetilde{C}^{\otimes n_2} \otimes \cdots \otimes \widetilde{C}^{\otimes n_s}$. The s-iterated product $\mu^{(s)}_{\Omega_{(12)}}$ permutes these s blocks as the permutation $\begin{pmatrix} 1 & 2 & \cdots & s-1 & s \\ s & s-1 & \cdots & 2 & 1 \end{pmatrix}$ in $\mathbb{S}_s$, where $\mathbb{S}_s$ denotes the symmetric group on s objects. The two first terms we obtain are

$$O_2 = F^2 - \tau F^2 = E^{1,1} - \tau E^{1,1}$$

and

$$\begin{aligned} O_3 &= (1+\tau_3)F^3 + \tau^{2,1}(F^2 \otimes 1)F^2 + \tau^{1,2}(1 \otimes F^2)F^2 \\ &= (\tau^{1,2} - 1 - \tau_3)(1 \otimes E^{1,1})E^{1,1} + \tau^{2,1}(E^{1,1} \otimes 1)E^{1,1} + (1+\tau_3)E^{2,1}, \end{aligned}$$

[1] We can also consider $\mu_\Omega(S \otimes 1)\nabla = \eta\epsilon$. It gives equivalent F^n but with a more complicated description because of the apparition of permutations. For example $F^3 = -(E^{1,1} \otimes 1)E^{1,1} - (\tau \otimes 1)E^{2,1}$.

where the permutations are $\tau^{1,2}(a\otimes b\otimes c) = \pm b\otimes c\otimes a$, $\tau^{2,1}(a\otimes b\otimes c) = \pm c\otimes a\otimes b$ and $\tau_3(a\otimes b\otimes c) = \pm c\otimes b\otimes a$; the signs being given by the Koszul sign rule.
We conclude,

Proposition 3.3. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra.*

1. *The cobar construction ΩC is an involutive DG-Hopf algebra if and only if all the obstructions $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$ defined in (3.5) for $n \geq 2$ are zero.*
2. *Let C be 2-reduced. If all the obstructions $O_n : \widetilde{C} \to \widetilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra whose the BV-operator is the Connes-Moscovici operator defined in (2.4).*

In fact we can make more precise the second point of the previous proposition. Let M be a DG-module and let $M_{\leq n}$ be the sub-module of elements $m \in M$ of degree $|m| \leq n$. The homotopy G-coalgebra structure on a connected coalgebra C preserves the filtration $C_{\leq n}$. Indeed the degree of the operations $E^{k,1} : \widetilde{C} \to \widetilde{C}^{\otimes k} \otimes \widetilde{C}$ is 0. We have,

Proposition 3.4. *Let $(C, d, \nabla_C, E^{k,1})$ be an i-reduced homotopy G-coalgebra, $i \geq 2$. If there exists an integer n such that $O_k = 0$ for $ki \leq n-1$, then $\Omega^2(C_{\leq n})$ is a homotopy BV-algebra.*

Proof. The degree of $O_k : \widetilde{C} \to \widetilde{C}^{\otimes k}$ is 0. Then on $s^{-1}(\overline{C}_{\leq n}) = \widetilde{C}_{\leq n-1}$ the only eventually non-zero operations $O_k : \widetilde{C}_{\leq n} \to (\widetilde{C}_{\leq n})^{\otimes k}$ are those with $ki \leq n-1$. ■

Remark 3.5. The two previous propositions work for a Hirsch coalgebra C instead of a homotopy G-coalgebra : the operations from (3.5) have the same definition in terms of F_n; only the F_n's defining the antipode differ. However, the involved techniques above are quite similar for Hirsch coalgebras.

Remark 3.6. For a general Hirsch coalgebra $(C, E^{i,j})$ the condition of cocommutativity of the coproduct on ΩC is $E^{i,j} = \tau E^{j,i}$ for all (i,j). Accordingly, with Proposition 1.4 we see (already in component three) that condition for the antipode to be involutive is weaker than condition for coproduct to be cocommutative. When the Hirsch coalgebra is a homotopy G-coalgebra, the cocommutativity of the coproduct means that the homotopy G-coalgebra is quasi trivial: $E^{1,1} = \tau E^{1,1}$ and $E^{k,1} = 0$ for $k \geq 2$.

4. Applications

4.1. On a homotopy G-coalgebra structure of $C_*(X)$

The chain complex $C_*(X)$ of a topological space or a simplicial set X has a rich algebraic structure, it is an E_∞-coalgebra. This was treated in many papers including J. McClure and J. Smith [16], C. Berger and B. Fresse [5]. For example, the surjection operad χ introduced in [16] acts on the normalized chain complex $C_*(X)$ of a simplicial set X making it into a coalgebra over χ, see [5, 16]. It has a filtration of suboperads

$$F_1\chi \subset F_2\chi \subset \cdots \subset F_{n-1}\chi \subset F_n\chi \subset \cdots \subset \chi.$$

This structure leads to a coproduct $\nabla_0 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$ first defined by Baues in [4]. The third stage filtration $F_3\chi$ gives a homotopy cocommutativity $\nabla_1 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$ to Baues' coproduct, see [4]. In turn the operation $\nabla_1 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$ is cocommutative up to a homotopy $\nabla_2 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$ and so on. The resulting structure is known as a structure of DG-bialgebra with Steenrod coproduct ∇_i. This was achieved by Kadeishvili in [10] where the corresponding operations in χ are given.

Baues' coproduct [3, p.334], [4, (2.9) equation (3)] on the cobar construction $\Omega C_*(X)$ corresponds to a homotopy G-coalgebra on $C_*(X)$. By a direct comparison, we see that this homotopy G-coalgebra structure coincides with the one given in [5, 16]. To be more precise, let

$$E_0^{k,1} : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes k} \otimes \widetilde{C}_*(X)$$

be the operations defined by Baues' coproduct

$$\nabla_0 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X).$$

Let

$$E^{1,k} : \widetilde{C}_*(X) \to \widetilde{C}_*(X) \otimes \widetilde{C}_*(X)^{\otimes k}$$

the operations defined by

$$E^{1,k}(\mathrm{s}^{-1}\sigma) = \sum_{0\leq j_1<j_2<\ldots<j_{2k}\leq n}$$
$$\pm \mathrm{s}^{-1}\sigma(0,\ldots,j_1,j_2,\ldots,j_3,j_4,\cdots,j_{2k-1},j_{2k},\ldots,n)\otimes \mathrm{s}^{-1}\sigma(j_1,\ldots,j_2)\otimes \mathrm{s}^{-1}\sigma(j_3,\ldots,j_4)\otimes\cdots\otimes \mathrm{s}^{-1}\sigma(j_{2k-1},\ldots j_{2k}),$$

for $\mathrm{s}^{-1}\sigma \in \widetilde{C}_n(X)$. These are the operations denoted by $AW(1,2,1,3,...,k-1,1,k)$ in [5, section 2.2] where $AW(u) : \widetilde{C}_*(X) \to \widetilde{C}_*(X)^{\otimes n}$ is defined for a surjection $u \in \chi(n)_d$. The operad $F_2\chi$ is generated by the surjections $(1,2)$ and $(1,2,1,3,...,1,k,1,k+1,1)$ for $k \geq 1$. Then the operations

$$E^{1,k} := AW(1,2,1,3,...,k,1,k+1,1),$$

define a homotopy G-coalgebra structure on $C_*(X)$. We have

Proposition 4.1. *Let X be a 1-reduced simplicial set. Then*

$$E_0^{k,1} = \pm\tau^{1,k}E^{1,k},$$

where $\tau^{1,k}$ orders the factors as the following permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 \\ 2 & 3 & \cdots & k+1 & 1 \end{pmatrix}.$$

Proof. First we recall Baues' coproduct ∇_0. We adopt the same conventions as in [3]. For $\sigma_i \in X$, the tensor $[\sigma_1|\sigma_2|...|\sigma_n]$ is the tensor $s^{-1}\sigma_{i_1}\otimes s^{-1}\sigma_{i_2}\otimes...\otimes s^{-1}\sigma_{i_r}$ where the indices i_j are such that $\sigma_{i_j} \in X_{n_{i_j}}$ with $n_{i_j} \geq 2$. For a subset $b = \{b_0 < b_1 < ... < b_r\} \subset \{0,1,...,n\}$ we denote by i_b the unique order-preserving injective function

$$i_b : \{0,1,...,r\} \to \{0,1,...,n\}$$

such that $\mathrm{Im}(i_b) = b$. Let $\sigma \in X_n$ and $0 \leq b_0 < b_1 \leq n$. We denote by $\sigma(b_0,...,b_1)$ the element $i_b^*\sigma \in X_{b_1-b_0}$ where $b = \{b_0, b_0+1,...,b_1-1,b_1\}$. Let $b \subset \{1,...,n-1\}$, we denote by $\sigma(0,b,n)$ the element $i_{b'}^*\sigma$ where $b' = \{0\}\cup b\cup\{n\}$.
Baues' coproduct ∇_0 [3, p.334] is defined on $\sigma \in X_n$, $n \geq 2$ by:

$$\begin{aligned} \nabla_0 : \Omega C_*(X) &\to \Omega C_*(X)\otimes\Omega C_*(X) \\ s^{-1}\sigma = [\sigma] &\mapsto \sum_{\substack{b=\{b_1<b_2<\ldots<b_r\}\\ b\subset\{1,\ldots,n-1\}}} (-1)^{\zeta}[\sigma(0,...,b_1)|\sigma(b_1,...,b_2)|...|\sigma(b_r,...,n)]\otimes[\sigma(0,b,n)] \end{aligned} \qquad (4.1)$$

where

$$\zeta = r|\sigma(0,..,b_1)| + \sum_{i=2}^{r}(r+1-i)|\sigma(b_{i-1},...,b_i)| - r(r+1)/2,$$

it is extended as an algebra morphism on the cobar construction.

We show that the coproduct ∇_0 defines the same homotopy G-coalgebra structure on $C_*(X)$ when X is 1-reduced. Let σ be an n-simplex, then Baues' coproduct is a sum over the subset $b = \{b_1 < b_2 < ... < b_r\} \subset \{1,...,n-1\}$. For such a b, we set $b_0 := 0$, $b_{r+1} := n$ and we defined $\beta_l \in b \cup \emptyset$ as follows. For $l = 1$,

$$\beta_1 := \min_{1\leq i\leq r+1}\{b_i \mid b_i - b_{i-1} \geq 2\},$$

and let η_1 be the index such that $b_{\eta_1} = \beta_1$; for $l \geq 2$,

$$\beta_l := \min_{\eta_{l-1}+1\leq i\leq r+1}\{b_i \mid b_i - b_{i-1} \geq 2\}$$

where η_l is the index such that $b_{\eta_l} = \beta_l$. Moreover, we set $\alpha_l := b_{\eta_l - 1}$. Let k be the integer such that $1 \leq l \leq k$. Explicitly, k is given by:

$$k = \#\{1 \leq i \leq r+1 \mid b_i - b_{i-1} \geq 2\}.$$

Thus the coproduct ∇_0 is

$$\nabla_0([\sigma]) = \sum_{k=0}^{n} \sum_{\substack{0\leq\alpha_1\leq\beta_1\leq\alpha_2\leq\cdots\leq\alpha_k\leq\beta_k\leq n \\ \beta_i-\alpha_i\geq 2,\ 1\leq i\leq k}} \pm\, s^{-1}\sigma(\alpha_1,...,\beta_1)\otimes s^{-1}\sigma(\alpha_2,...,\beta_2)\otimes\cdots$$
$$\cdots\otimes s^{-1}\sigma(\alpha_k,...,\beta_k)\otimes s^{-1}\sigma(0,...,\alpha_1,\beta_1,...,\alpha_2,\cdots,\alpha_k,\beta_k,...,n).$$

Thus we have,

$$E^{k,1}(s^{-1}\sigma) = \sum \pm\, s^{-1}\sigma(\alpha_1,...,\beta_1)\otimes s^{-1}\sigma(\alpha_2,...,\beta_2)\otimes\cdots\otimes s^{-1}\sigma(\alpha_k,...,\beta_k)\otimes$$
$$s^{-1}\sigma(0,...,\alpha_1,\beta_1,...,\alpha_2,\cdots,\alpha_k,\beta_k,...,n).$$

The result is obtained by setting $j_{2l-1} = \alpha_l$ and $j_{2l} = \beta_l$. Since X is 1-reduced, the elements $\sigma(j_l,...,j_{l+1})$ such that $j_{l+1} - j_l = 1$ are elements in $X_1 = * = s_0(*)$ and then are degenerate. ∎

In the two next subsections 4.2 and 4.3 we adopt the above notations for the homotopy G-coalgebra structure $E^{1,k}$ on $C_*(X)$.

4.2. *Obstruction to the involutivity of the antipode of* $\Omega C_*(\Sigma X)$

Let ΣX be a simplicial suspension of a simplicial set X. We show that the family of obstructions $O_n : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X)^{\otimes n}$ defined in (3.5) can be reduced to $O_2 : \widetilde{C}_*(\Sigma X) \to \widetilde{C}_*(\Sigma X)^{\otimes 2}$; the latter being governed by the (lack of) cocommutativity of the operation $E^{1,1}$.

Definition 4.2. [15, Definition 27.6 p.124] Let X be a simplicial set such that $X_0 = *$ and with face and degeneracy operators $s_j : X_n \to X_{n+1}$ and $d_j : X_n \to X_{n-1}$. The **simplicial suspension** ΣX is defined as follow. The component $(\Sigma X)_0$ is just an element a_0 and $(\Sigma X)_n = \{(i,x) \in \mathbb{N}^{\geq 1} \times X_{n-i}\}/\Big((i, s_0^n(*)) = s_0^{n+i}(a_0)\Big)$. We set $a_n = s_0^n(a_0)$. The face and degeneracy operators are generated by:

- $d_0(1,x) = a_n$ for all $x \in X_n$;

- $d_1(1,x) = a_0$ for all $x \in X_0$;
- $d_{i+1}(1,x) = (1, d_i(x))$ for all $x \in X_n, n > 0$;
- $s_0(i,x) = (i+1, x)$;
- $s_{i+1}(1,x) = (1, s_i(x))$,

with the other face and degeneracy operators determined by the requirement that ΣX is a simplicial set.

Proposition 4.3. *[9] The differential of $(\Omega C_*(\Sigma X), d_0 + d_1)$ is reduced to its linear part d_0.*

Proof. The only non degenerate elements in ΣX are $(1,x)$ and we have that $d_0(1,x) = a_n$ is degenerate. Therefore the Alexander-Whitney coproduct on $C_*(\Sigma X)$ is primitive (we recall that C_* are the normalized chains). Then, the reduced coalgebra $\overline{C}_*(\Sigma X)$ is a trivial coalgebra; so the quadratic part d_1 of the differential $d_\Omega = d_0 + d_1$ on the cobar construction is trivial. ■

A natural coproduct to define is the shuffle coproduct (primitive on cogenerators) which gives a cocommutative DG-Hopf structure to $\Omega C_*(\Sigma X)$; applying Proposition 2.5 we obtain on $\Omega^2 C_*(\Sigma X)$ a homotopy BV-algebra structure. However, this coproduct does not correspond to Baues' coproduct ∇_0. Indeed, the homotopy G-algebra structure on $C_*(\Sigma X)$ is not completely trivial. Because of the cocommutativity of the Alexander-Whitney coproduct on $C_*(\Sigma X)$, the operation $E^{1,1}$ must be a chain map but not necessary the zero map. We obtain,

Proposition 4.4. *The homotopy G-coalgebra structure $E^{k,1}$ on $C_*(\Sigma X)$ given by Baues' coproduct is*

$$E^{1,1}(\mathrm{s}^{-1}\sigma) = \sum_{1<l<n} \pm \mathrm{s}^{-1}\sigma(0,...,l) \otimes \mathrm{s}^{-1}\sigma(0,l,l+1,...,n);$$

$$E^{k,1} = 0 \ \textit{for } k \geq 2.$$

Proof. For a non-degenerate $\sigma \in (\Sigma X)_n$, we have $d_0\sigma = \sigma(1,...,n)$ is degenerate. Consequently, the operation

$$E^{1,1}(\mathrm{s}^{-1}\sigma) = \sum_{k<l} \pm \mathrm{s}^{-1}\sigma(k,...,l) \otimes \mathrm{s}^{-1}\sigma(0,...,k,l,l+1,...,n)$$

reduces to

$$E^{1,1}(\mathrm{s}^{-1}\sigma) = \sum_{1<l<n} \pm \mathrm{s}^{-1}\sigma(0,...,l) \otimes \mathrm{s}^{-1}\sigma(0,l,l+1,...,n).$$

The higher operations $E^{k,1}$ for $k \geq 2$ given by

$$E^{1,k}(\mathrm{s}^{-1}\sigma) = \sum_{0\leq j_1<j_2<...<j_{2k}\leq n}$$
$$\pm \mathrm{s}^{-1}\sigma(0,...,j_1,j_2,...,j_3,j_4,\cdots,j_{2k-1},j_{2k},...,n) \otimes \mathrm{s}^{-1}\sigma(j_1,...,j_2) \otimes \mathrm{s}^{-1}\sigma(j_3,...,j_4) \otimes \cdots \otimes \mathrm{s}^{-1}\sigma(j_{2k-1},...j_{2k})$$

are zero since the terms $\sigma(j_3,...,j_4)$ are degenerate. ■

The triviality of higher operations $E^{k,1}$ for $k \geq 2$ does not imply the vanishing of higher obstructions O_n since the $E^{1,1}$ operation appears in all the O_n's. However, we can reduce the family of obstructions O_n to only O_2 in this case.

Proposition 4.5. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra with $E^{k,1} = 0$ for $k \geq 2$. If $O_2 = E^{1,1} - \tau E^{1,1}$ is zero, then so is O_n for $n \geq 2$.*

Proof. By making explicit the coassociativity of the coproduct $\nabla : \Omega C \to \Omega C$ we obtain, in particular, the equation

$$(1 \otimes E^{1,1})E^{1,1} - (E^{1,1} \otimes 1)E^{1,1} = (\tau \otimes 1)E^{2,1} + (1 \otimes 1)E^{2,1}.$$

Therefore the triviality of the higher operation $E^{2,1}$ implies that $E^{1,1}$ is coassociative. Together with the vanishing of O_2 we obtain a coassociative and cocommutative operation $E^{1,1}$. We use this fact to vanish the O_n's : by (3.4), the terms F^i, $i \leq n$ involved in O_n are

$$F^n = (-1)^n (1^{\otimes n-2} \otimes E^{1,1}) \cdots (1 \otimes E^{1,1})E^{1,1}.$$

Now because of the coassociativity of $E^{1,1}$ we can write each term $\mu^{(s)}_{\Omega_{(12)}}(F^{n_1} \otimes ... \otimes F^{n_s})F^s$ of the sum

$$O_n = \sum_{s=1}^{n} \sum_{n_1+...+n_s=n} \mu^{(s)}_{\Omega_{(12)}}(F^{n_1} \otimes ... \otimes F^{n_s})F^s$$

as $\pm\sigma \circ F^n$ where σ is a permutation (depending of the term we consider). Now using the cocommutativity and again the coassociativity of $E^{1,1}$ we can remove all transpositions to obtain $\pm F^n$. A direct counting shows that positive and negative terms are equal in number : the sign of $(F^{n_1} \otimes ... \otimes F^{n_s})F^s$ is $(-1)^{n_1+...+n_s+s} = (-1)^{n+s}$ and the number of partitions of n into s integers ≥ 1 is $\binom{n-1}{s-1}$. ■

Therefore we can consider O_2 as the only obstruction to the involutivity of the antipode on the cobar construction.

4.3. Homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$

Here we prove that the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ is trivial and then that $\Omega C_*(\Sigma^2 X)$ is an involutive DG-Hopf algebra with the shuffle coproduct.

Theorem 4.6. *Let $\Sigma^2 X$ be a double suspension. Then :*

- *the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ corresponding to Baues' coproduct*

$$\nabla_0 : \Omega C_*(\Sigma^2 X) \to \Omega C_*(\Sigma^2 X) \otimes \Omega C_*(\Sigma^2 X),$$

  has trivial higher operations i.e. $E^{1,k} = 0$ for $k \geq 1$;

- *the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is a homotopy BV-algebra with the BV-operator from (2.4).*

Proof. By Proposition 4.4 the operations $E^{k,1}$ for $k \geq 2$ are trivial. Therefore to prove the first statement it remains to prove that $E^{1,1}$ is trivial. This follows that $d_1(1,(1,x)) = (1, d_0(1,x)) = (1, a_n) = s_{n+1}(1, a_0)$ is degenerate. Indeed, the operation

$$E^{1,1}(s^{-1}\sigma) = \sum_{1<l<n} \pm s^{-1}\sigma(0,...,l) \otimes s^{-1}\sigma(0,l,l+1,...,n)$$

is trivial since $l > 1$.
For the second statement since the homotopy G-coalgebra structure is trivial, all the obstructions O_n are zero and so, by Proposition 3.3, we obtain the announced homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$. ■

We make explicit the homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$. Let us first observe that Baues' coproduct ∇_0 is a shuffle coproduct. Indeed, the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ being trivial, for any element $a \in \overline{C}_*(\Sigma^2 X)$ we have

$$\nabla_0(\mathrm{s}^{-1} a) = (E_{1,0} + E_{0,1})(\mathrm{s}^{-1} a) = \mathrm{s}^{-1} a \otimes 1 + 1 \otimes \mathrm{s}^{-1} a \in \Omega C_*(\Sigma^2 X) \otimes \Omega C_*(\Sigma^2 X).$$

Therefore its extension $\nabla_0 : \Omega C_*(\Sigma^2 X) \to \Omega C_*(\Sigma^2 X) \otimes \Omega C_*(\Sigma^2 X)$ as algebra morphism is the shuffle coproduct, see [12, Theorem III.2.4]. We write ∇_0 as

$$\nabla_0([a_1|...|a_n]) = \sum_{I_0 \cup I_1} \pm [a_{I_0}] \otimes [a_{I_1}]$$

where the sum is taken over all partitions $I_0 \sqcup I_1$ of $I = \{1, ..., n\}$ with $I_0 = \{i_1 < i_2 < ... < i_k\}$ and $I_1 = \{j_1 < j_2 < ... < j_{n-k}\}$. We denote a_{I_0} to be $a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k}$. Also we denote $a_{I_0^{-1}}$ to be $a_{i_k} \otimes ... \otimes a_{i_2} \otimes a_{i_1}$ and similarly for a_{I_1}.

Now we make explicit the BV-operator on a 2 and 3-tensor. For the sake of simplicity we do not keep track of signs; also to avoid confusion we write $\underline{a} = [a_1|a_2|...|a_k]$ for the basic elements of $\Omega C_*(\Sigma^2 X)$ and $[\underline{a}_1, \underline{a}_2, ..., \underline{a}_n]$ for the basic elements of $\Omega^2 C_*(\Sigma^2 X)$.
Thus $[\underline{a}_1, \underline{a}_2, ..., \underline{a}_n]$ is $[[a_{1,1}|a_{1,2}|...|a_{1,k_1}], [a_{2,1}|a_{2,2}|...|a_{2,k_2}], ..., [a_{n,1}|a_{n,2}|...|a_{n,k_n}]]$.
With these conventions, the BV-operator Δ is given by

$$\begin{aligned}
\Delta([\underline{a}, \underline{b}]) = \Delta([[a_1|...|a_m], [b_1|...|b_n]]) &= [[a_m|...|a_1|b_1|...|b_n]] + [[b_n|...|b_1|a_1|...|a_m]].\\
\Delta([[a_1|...|a_m], [b_1|...|b_n], [c_1|...|c_r]]) &= [[a_{I_1^{-1}}|b_1|...|b_n], [a_{I_0^{-1}}|c_1|...|c_r]]\\
&\quad + [[b_{I_1^{-1}}|c_1|...|c_r], [b_{I_0^{-1}}|a_1|...|a_m]]\\
&\quad + [[c_{I_1^{-1}}|a_1|...|a_m], [c_{I_0^{-1}}|b_1|...|b_n]].
\end{aligned}$$

The homotopy H from Proposition 2.5 is given by

$$H([\underline{a}, \underline{b}]) = 0.$$

To know explicitly each term of

$$\Delta([\underline{a}, \underline{b}, \underline{c}]) - [\underline{a}, \underline{b}, \Delta([\underline{c}])] - [\Delta([\underline{a}, \underline{b}]), \underline{c}] - \{[\underline{a}, \underline{b}]; [\underline{c}]\} = \partial H([\underline{a}, \underline{b}]; [\underline{c}])$$

we need to know the following terms:

$$H([\underline{a}, \underline{b}]; [\underline{c}]) = H([[a_1|...|a_m], [b_1|...|b_n]]; [[c_1|...|c_r]]) = \pm [[b_n|...|b_1|a_1|...|a_m|c_1|...|c_r]],$$

$$\begin{aligned}
H([\underline{a}, \underline{b}, \underline{c}]; [\underline{d}]) &= H([[a_1|...|a_m], [b_1|...|b_n], [c_1|...|c_r]]; [[d_1|...|d_s]])\\
&= \pm [[c_{I_1^{-1}}|a_1|...|a_m|d_1|...|d_s], [c_{I_0^{-1}}|b_1|...|b_n]]\\
&\quad \pm [[c_{I_1^{-1}}|a_1|...|a_m], [c_{I_0^{-1}}|b_1|...|b_n|d_1|...|d_s]]\\
&\quad \pm [[b_{I_1^{-1}}|c_1|...|c_r], [b_{I_0^{-1}}|c_1|...|c_r|d_1|...|d_s]],
\end{aligned}$$

and

$$H([\underline{a}, \underline{b}]; [\underline{c}, \underline{d}]) = \pm [[c_{I_1^{-1}}|a_1|...|a_m|d_1|...|d_s], [c_{I_0^{-1}}|b_1|...|b_n]].$$

Remark 4.7. From a topological point of view, let us consider X to be a connected (countable) CW-complex with one vertex. The James/Milgram's models $J_i(X)$ are H-spaces homotopically equivalent to $\Omega^i \Sigma^i X$, see [18, theorem 5.2]. Moreover, for $i \geq 1$ the cellular chain complex $C_*(J_i(\Sigma X))$ is a cocommutative, primitively generated DG-Hopf algebra, and $C_*(J_{i+1}(X))$ is isomorphic to $\Omega C_*(J_i(\Sigma X))$, [18, theorem 6.1 and 6.2]. Then using Proposition 2.5 we get a homotopy BV-algebra structure on it which is similar to the one obtained in the simplicial context. Also, we have a DGA-quasi-isomorphism

$$\Omega C_*(J_1(\Sigma X)) \longrightarrow C_*(J_2(X)) \xrightarrow{C_*(j_2)} C_*(\Omega^2\Sigma^2 X).$$

4.4. Homotopy BV-algebra structure on $\Omega^2 C_*(X)$ over $R \supset \mathbb{Q}$

In [4], Baues gives an explicit construction of the cobar construction of a 1-reduced simplicial set X over the integer coefficient ring as a cocommutative up to homotopy DG-bialgebra. By using methods of [2], he shows that over a ring R containing $\mathbb{Q}$ as a subring, this DG-bialgebra can be deformed into a strictly cocommutative DG-bialgebra [4, Theorem 4.7]. The latter is isomorphic as DG-Hopf algebras to the universal enveloping algebra of the Lie algebra $L(s^{-1}\overline{C}_*X)$ generated by the desuspension of the reduced coalgebra of normalized chain complex, combine [4, Theorem 4.8] and [12, Proposition V.2.4]. Hence, over such a ring R, the cobar construction $\Omega C_*(X)$ is (can be deformed into) an involutive DG-Hopf algebra. By applying Proposition 2.5 to the resulting involutive cobar construction $(\Omega C_*(X), \nabla_0', S')$, we obtain

Theorem 4.8. *Let X be a 2-reduced simplicial set. Then the double cobar construction $\Omega(\Omega C_*(X), \nabla_0', S')$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator the Connes-Moscovici operator.*

In the sequel we detail how the antipode is deformed. We extend the deformation (given in [4]) of the DG-bialgebra structure of the cobar construction to a deformation of the DG-Hopf structure. The resulting cobar construction $(\Omega C_*(X), \nabla_0', S')$ comes with (anti)derivation homotopies (see below) connecting the obtained DG-Hopf structure (∇_0', S') with the initial one (∇_0, S).

First of all we extend some definitions from [4].

Let $\mathbf{DGA}_0$ be the category of connected DG-algebras (associative). For a free DG-module V, we denote by $L(V)$ the DG-Lie algebra which is the free graded Lie algebra on the free module V. We set $V_{\leq n}$ (resp. $V_{<n}$) the sub DG-module of elements $v \in V$ such that $|v| \leq n$ (resp. $|v| < n$). Similarly, $V_{\geq n}$ (resp. $V_{>n}$) denotes the subset of elements $v \in V$ such that $|v| \geq n$ (resp. $|v| > n$).

Definition 4.9. Let $f, g : A \to B$ be two maps between two DG-modules A, B. A **derivation homotopy** between f and g is a map $F : A \to B$ satisfying

$$dF + Fd = f - g \tag{4.2}$$

$$F(ab) = F(a)g(b) + (-1)^{|F||a|} f(a)F(b) \ \ \forall a, b \in A. \tag{4.3}$$

An **antiderivation homotopy** between f and g is a map $\Gamma : A \to B$ satisfying

$$d\Gamma + \Gamma d = f - g \tag{4.4}$$

$$\Gamma(ab) = (-1)^{|a||b|}(\Gamma(b)g(a) + (-1)^{|\Gamma||a|} f(b)\Gamma(a)) \ \ \forall a, b \in A. \tag{4.5}$$

Definition 4.10. A **homotopy DG-bialgebra** (A, ∇, G_1, G_2) is an object A in $\mathbf{DGA}_0$ together with a coproduct ∇ in $\mathbf{DGA}_0$ making (A, ∇) into a coalgebra in $\mathbf{DGA}_0$ cocommutative up to G_1, coassociative up to G_2, both being derivation homotopies.

Definition 4.11. Let (A, ∇, G_1, G_2) be a homotopy DG-bialgebra such that $A = TV$, V being a DG-module with $V_0 = 0$. The counit is the augmentation of TV. We call (A, ∇, G_1, G_2, V) n**-good** if the following conditions (4.6), (4.7),(4.8) hold.

$$\nabla = \tau\nabla \text{ and } (\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla \text{ on } V_{\leq n} \tag{4.6}$$

$$d(V_{\leq n+1}) \subset L(V_{\leq n}) \subset T(V) = A \tag{4.7}$$

$$V_{\leq n} \subset \ker(\overline{\nabla}). \tag{4.8}$$

If moreover there exists a chain map $S : A \to A$ such that the condition:

$$\mu(1 \otimes S)\nabla = \eta\epsilon = \mu(S \otimes 1)\nabla \text{ on } V_{\leq n} \tag{4.9}$$

is satisfied, then A is called n-**good**$^+$.

Lemma 4.12. *Let $\mathcal{A} := (A = TV, \nabla, G_1, 0, S, V)$ be a homotopy DG-bialgebra with antipode S. We suppose it is n-good$^+$ for a map $S : A \to A$. Then there exists a homotopy DG-bialgebra $(A, \nabla^{n+1}, G_1^{n+1}, G_2^{n+1}, S^{n+1}, V^{n+1})$ which both extends $\mathcal{A}$ and is $(n+1)$-good$^+$. Moreover, there is a derivation homotopy $F^{n+1} : \nabla \simeq \nabla^{n+1}$ and an antiderivation homotopy $\Gamma^{n+1} : S \simeq S^{n+1}$ with $F^{n+1}(a) = 0$ and $\Gamma^{n+1}(a) = 0$ for $|a| < n$, $a \in A$.*

Proof. The part $(n+1)$-good as bialgebra is already done in [4, Theorem 4.5] where ∇^{n+1}, G_1^{n+1},G_2^{n+1}, F^{n+1} and V^{n+1} are defined. Therefore we only need to construct S^{n+1} and Γ^{n+1}.
We recall that $(V^{n+1})_k = V_k$ for $k \neq n+1, n+2$. Let $S^{n+1} = S - R^{n+1}$ where $R^{n+1} = \mu(1 \otimes S)\nabla^{n+1} - \eta\epsilon$. Then $R^{n+1} = 0$ on $V_{\leq n}$. On $(V^{n+1})_{n+1}$ we have

$$\begin{aligned}
\mu(1 \otimes S^{n+1})\nabla^{n+1} &= \mu(1 \otimes S)\nabla^{n+1} - \mu(1 \otimes R^{n+1})\nabla^{n+1} \\
&= \mu(1 \otimes S)\nabla^{n+1} - R^{n+1} \\
&= \mu(1 \otimes S)\nabla^{n+1} - \mu(1 \otimes S)\nabla^{n+1} + \eta\epsilon \\
&= \eta\epsilon.
\end{aligned}$$

The homotopy F^{n+1} between ∇ and ∇^{n+1} is such that $F^{n+1} = 0$ on $(V^{n+1})_{<n} \cup (V^{n+1})_{>n+1}$. Setting $\Gamma^{n+1} = -\mu(1 \otimes S)F^{n+1}$ we obtain the desired homotopy. Indeed,

$$d\Gamma^{n+1} + \Gamma^{n+1}d = -\mu(1 \otimes S)(dF^{n+1} + F^{n+1}d) = \mu(1 \otimes S)(\nabla^{n+1} - \nabla) = R^{n+1} = S - S^{n+1}.$$

And Γ^{n+1} is such that $\Gamma^{n+1} = 0$ on $(V^{n+1})_{<n} \cup (V^{n+1})_{>n+1}$.
Next we extend S^{n+1} on $T((V^{n+1})_{\leq n+1})$ as an algebra antimorphism and we extend Γ^{n+1} as an antiderivation homotopy. We have $\mu(S^{n+1} \otimes 1)\nabla^{n+1} = \eta\epsilon$ on $(V^{n+1})_{\leq n+1}$ since ∇^{n+1} is coassociative on $(V^{n+1})_{\leq n+1}$. ■

Lemma 4.13. *Let $(A = TV, \nabla, G_1, S)$ be a DG-Hopf algebra cocommutative up to the homotopy G_1 and with S as antipode. Suppose that (A, ∇, G_1, S, V) is 1-good$^+$. Then there is a coproduct ∇' and an antipode S' such that (A, ∇', S') is a cocommutative DG-Hopf algebra. Moreover there is a derivation homotopy $F : \nabla \simeq \nabla'$ and an antiderivation homotopy $\Gamma : S \simeq S'$.*

Proof. An iteration of Lemma 4.12 yields, for each $n \geq 1$, an n-good$^+$ homotopy DG-bialgebra $(A, \nabla^{n+1}, G_1^{n+1}, G_2^{n+1}, S^{n+1}, V^{n+1})$. Hence we define $\nabla'(v) = \nabla^n(v)$ and $S'(v) = S^n(v)$ for $|v| = n$. We have $\mu(1 \otimes S')\nabla' = \eta\epsilon$. The derivation homotopy F is defined as $\sum_{n\geq 1} F^n$ and the antiderivation homotopy Γ is defined as $\sum_{n\geq 1} \Gamma^n$. ■

Proposition 4.14. *Let $\mathbb{Q} \subset R$ and let X be a 1-reduced simplicial set. Then there is both a coproduct ∇' and an antipode S' on the cobar construction $\Omega C_* X$ such that $(\Omega C_*(X), \nabla', S')$ is a cocommutative DG-Hopf algebra. Moreover there is a derivation homotopy $F : \nabla_0 \simeq \nabla'$ and an antiderivation homotopy $\Gamma : S_0 \simeq S'$, where ∇_0 and S_0 are respectively Baues' coproduct [4, (3)] and the associated antipode.*

Proof. We denote by G_1 the homotopy to the cocommutativity of the coproduct ∇_0. It is defined in [4, (4)]. We apply Lemma 4.13 to $(\Omega C_*(X), \nabla_0, G_1, 0, S, s^{-1}\overline{C}_* X)$ which is 1-good$^+$. ■

Appendix

Here we recall and develop some facts about the Hirsch and the homotopy G-algebras. A Hirsch $(\Lambda, d_\Lambda, \cdot, \{E_{1,k}\}_{k\geq 1})$ corresponds to a product $\mu_E : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \to \mathcal{B}\Lambda$ such that $(\mathcal{B}\Lambda, d_{\mathcal{B}\Lambda}, \mu_E)$ is a unital DG-bialgebra.
We write down the relations among the $E_{i,j}$, $i, j \geq 1$ coming from both the associativity of μ_E and the Leibniz relation $d_{\mathcal{B}\Lambda}\mu_E = \mu_E(d_{\mathcal{B}\Lambda} \otimes 1 + 1 \otimes d_{\mathcal{B}\Lambda})$.
We detail the construction of the operation $E_{1,1}$ defined in (2.3).

Unit condition

For all $\underline{s\,a} = s\,a_1 \otimes ... \otimes s\,a_i \in B\Lambda$ we have:

$$\mu_E(1_\Lambda \otimes \underline{s\,a}) = \mu_E(\underline{s\,a} \otimes 1_\Lambda) = \underline{s\,a} \tag{4.10}$$

The product being determined by its projection on Λ we have:

$$pr\mu_E(1_\Lambda \otimes \underline{s\,a}) = \widetilde{E}_{0,i}(1_\Lambda \otimes \underline{s\,a}) = pr(s\,a_1 \otimes ... \otimes s\,a_i) = \begin{cases} a_1 & \text{if } \ i = 1; \\ 0 & \text{if } \ i \neq 1, \end{cases}$$

and also the symmetric relation. Thus,

$$E_{0,i} = E_{i,0} = 0 \ \text{ for all } \ i \neq 1 \qquad\qquad \text{and} \qquad\qquad E_{0,1} = E_{1,0} = Id_\Lambda. \tag{4.11}$$

Associativity condition

With the sign convention (2.1), the associativity of μ_E gives on $\Lambda^{\otimes i} \otimes \Lambda^{\otimes j} \otimes \Lambda^{\otimes k}$:

$$\begin{aligned}
&E_{1,k}(E_{i,j}(a_1,...,a_i;b_1,...b_j);c_1,...,c_k)+\\
&\qquad\sum_{n=1}^{i+j}\sum_{\substack{0\leq i_1\leq ...\leq i_n\leq i\\ 0\leq j_1\leq ...\leq j_n\leq j}}(-1)^{\alpha_1}E_{n+1,k}(E_{i_1,j_1}(a_1,...,a_{i_1};b_1,...b_{j_1}),...\\
&\qquad\qquad ...,E_{i-i_n,j-j_n}(a_{i_n+1},...,a_i;b_{j_n+1},...b_j);c_1,...,c_k)\\
&=\sum_{m=1}^{j+k}\sum_{\substack{0\leq j_1\leq ...\leq j_m\leq j\\ 0\leq k_1\leq ...\leq k_m\leq k}}(-1)^{\alpha_2}E_{i,m+1}(a_1,...,a_i;E_{j_1,k_1}(b_1,...,b_{j_1};c_1,...c_{k_1}),...\\
&\qquad\qquad ...,E_{j-j_m,k-k_m}(b_{j_m+1},...,b_j;c_{k_m+1},...c_k))\\
&\qquad\qquad\qquad + E_{i,1}(a_1,...,a_i;E_{j,k}(b_1,...,b_j;c_1,...c_k))
\end{aligned}\tag{4.12}$$

where

$$\alpha_1 = \sum_{u=1}^{n}\left(\sum_{s=i_u+1}^{i_{u+1}}|a_s| + i_{u+1} - i_u\right)\left(\sum_{s=1}^{j_u}|b_s| + j_u\right)$$

$$\alpha_2 = \sum_{u=1}^{m}\left(\sum_{s=j_u+1}^{j_{u+1}}|b_s| + j_{u+1} - j_u\right)\left(\sum_{s=1}^{k_u}|c_s| + k_u\right)$$

with

$$i_0 = 0; \quad j_0 = 0; \quad k_0 = 0;$$
$$i_{n+1} = i; \quad j_{n+1} = j_{m+1} = j; \quad k_{m+1} = k.$$

For $i = j = k = 1$, that is on $\Lambda \otimes \Lambda \otimes \Lambda$, the relation (4.12) gives:

$$E_{1,1}(E_{1,1}(a;b);c) = E_{1,1}(a;E_{1,1}(b;c)) + E_{1,2}(a;b,c) + (-1)^{(|b|-1)(|c|-1)}E_{1,2}(a;c,b) \\ - E_{2,1}(a,b;c) - (-1)^{(|a|-1)(|b|-1)}E_{2,1}(b,a;c). \quad (4.13)$$

Leibniz relation

On $\Lambda^{\otimes i} \otimes \Lambda^{\otimes j}$, the projection of $d_{\mathcal{B}}\mu_E = \mu_E(d_{\mathcal{B}} \otimes 1 + 1 \otimes d_{\mathcal{B}})$ gives:

$$\begin{aligned} d_\Lambda E_{i,j}(a_1, ..., a_i; b_1, ..., b_j) \\ + \sum_{\substack{0 \le i_1 \le i \\ 0 \le j_1 \le j}} (-1)^{\beta_2} E_{i_1,j_1}(a_1, ..., a_{i_1}; b_1, ..., b_{j_1}) \cdot E_{i-i_1,j-j_1}(a_{i_1+1}, ..., a_i; b_{j_1+1}, ..., b_j) = \\ = \sum_{l=1}^{i} (-1)^{\beta_3} E_{i,j}(a_1, ..., d_\Lambda(a_l), ..., a_i; b_1, ..., b_j) \\ + \sum_{l=1}^{i-1} (-1)^{\beta_4} E_{i-1,j}(a_1, ..., a_l \cdot a_{l+1}, ..., a_i; b_1, ..., b_j) \\ + \sum_{l=1}^{j} (-1)^{\beta_5} E_{i,j}(a_1, ..., a_i; b_1, ..., d_\Lambda(b_l), ..., b_j) \\ + \sum_{l=1}^{j-1} (-1)^{\beta_6} E_{i,j-1}(a_1, ..., a_i; b_1, ..., b_l \cdot b_{l+1}, ..., b_j) \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} \beta_2 &= \sum_{s=1}^{i_1} |a_s| + \sum_{s=1}^{j_1} |b_s| + i_1 + j_1 + \left(\sum_{s=i_1+1}^{i} |a_s| + i - i_1 \right) \left(\sum_{s=1}^{j_1} |b_s| + j_1 \right) \\ \beta_3 &= \eta_0(\underline{\mathrm{s}\, a}) \\ \beta_4 &= \eta_1(\underline{\mathrm{s}\, a}) \\ \beta_5 &= \sum_{s=1}^{i} |a_s| + i + \eta_0(\underline{\mathrm{s}\, b}) \\ \beta_6 &= \sum_{s=1}^{i} |a_s| + i + \eta_1(\underline{\mathrm{s}\, b}). \end{aligned}$$

For $\underline{\mathrm{s}\, a} := \mathrm{s}\, a_1 \otimes \cdots \otimes \mathrm{s}\, a_i$, the following signs

$$\eta_0(\underline{\mathrm{s}\, a}) = \sum_{s=1}^{l-1} |a_s| + l$$

$$\eta_1(\underline{\mathrm{s}\, a}) = \sum_{s=1}^{l} |a_s| + l,$$

are the signs of the differential of the bar construction. And similarly for $\underline{\mathrm{s}\, b} = \mathrm{s}\, b_1 \otimes \cdots \otimes \mathrm{s}\, b_j$.

Remark 4.15. When the twisting cochain $\widetilde{E} : \mathcal{B}\,\Lambda \otimes \mathcal{B}\,\Lambda \to \Lambda$ has $E_{i,j} = 0$ except $E_{1,0}$ and $E_{0,1}$, then Λ is a commutative DG-algebra.

We recall (cf. Definition 2.2) that when the twisting cochain $\widetilde{E} : \mathcal{B}\,\Lambda \otimes \mathcal{B}\,\Lambda \to \Lambda$ satisfies $\widetilde{E}_{i,j} = 0$ for $i \ge 2$, then $(\Lambda, d, \mu_\Lambda, \{E_{1,j}\})$ is called a homotopy G-algebra.

Remark 4.16. [13, Proposition 3.2] The condition $E_{i,j} = 0$ for $i \ge 2$ is equivalent to the following condition: for each integer r, $I_r := \oplus_{n \ge r}(\mathrm{s}^{-1}\,\overline{\Lambda})^{\otimes n}$ is a right ideal for the product $\mu_E : \mathcal{B}\,\Lambda \otimes \mathcal{B}\,\Lambda \to \mathcal{B}\,\Lambda$.

For a homotopy G-algebra Λ, the equation (4.14) gives the three following equalities.

On $\Lambda^{\otimes 1}\otimes\Lambda^{\otimes 1}$:

$$d_\Lambda E_{1,1}(a;b)-E_{1,1}(d_\Lambda a;b)+(-1)^{|a|}E_{1,1}(a;d_\Lambda b)=(-1)^{|a|}\Big(a\cdot b-(-1)^{|a||b|}b\cdot a\Big). \tag{4.15}$$

On $\Lambda^{\otimes 2}\otimes\Lambda^{\otimes 1}$:

$$E_{1,1}(a_1\cdot a_2;b)=a_1\cdot E_{1,1}(a_2;b)+(-1)^{|a_2|(|b|-1)}E_{1,1}(a_1;b)\cdot a_2. \tag{4.16}$$

On $\Lambda^{\otimes 1}\otimes\Lambda^{\otimes 2}$:

$$\begin{aligned}d_\Lambda E_{1,2}(a;b_1,b_2)+E_{1,2}(d_\Lambda a;b_1,b_2)+(-1)^{|a|+1}E_{1,2}(a;d_\Lambda b_1,b_2)+(-1)^{|a|+|b_1|}E_{1,2}(a;b_1,d_\Lambda b_2)\\ =(-1)^{|a|+|b_1|+1}\Big(E_{1,1}(a;b_1)b_2+(-1)^{(|a|-1)|b_1|}b_1E_{1,1}(a;b_2)-E_{1,1}(a;b_1b_2)\Big).\end{aligned} \tag{4.17}$$

The sign in (2.3)

Now we give the construction of the operation $E_{1,1}$ in (2.3). Recall that B is a DG-bialgebra and that $E_{1,1}:\Lambda\otimes\Lambda\to\Lambda$ with $\Lambda:=\Omega B$.
First, we set $E_{1,1}(\mathrm{s}^{-1}a;\mathrm{s}^{-1}b):=\mathrm{s}^{-1}(a\cdot b)$ for all $a,b\in B$. Next, using the equation (4.16), we extend this to:

$$E_{1,1}(\mathrm{s}^{-1}a_1\otimes\ldots\otimes\mathrm{s}^{-1}a_m;\mathrm{s}^{-1}b):=\sum_{l=1}^{m}(-1)^{\gamma_1}\,\mathrm{s}^{-1}a_1\otimes\ldots\otimes\mathrm{s}^{-1}(a_l\cdot b)\otimes\mathrm{s}^{-1}a_{l+1}\otimes\ldots\otimes\mathrm{s}^{-1}a_m,$$

for all homogeneous elements $a_1,a_2,\cdots,a_m,b\in B$, where

$$\gamma_1=|b|\left(\sum_{s=l+1}^{m}|a_s|-m+l\right).$$

On the other hand, using a slight abuse of notation, we set

$$E_{1,1}(\mathrm{s}^{-1}a;\mathrm{s}^{-1}b_1\otimes\mathrm{s}^{-1}b_2):=(-1)^{|a^2||b_1|+|a^2|}\,\mathrm{s}^{-1}(a^1\cdot b_1)\otimes\mathrm{s}^{-1}(a^2\cdot b_2).$$

The abuse of notation comes from the fact that the terms $(-1)^{|a||b_1|+|a|}\,\mathrm{s}^{-1}(b_1)\otimes\mathrm{s}^{-1}(a\cdot b_2)$ and $\mathrm{s}^{-1}(a\cdot b_1)\otimes\mathrm{s}^{-1}(b_2)$ belong to the above terms i.e. the coproduct of B evaluated on the element a is not reduced.
Using the equation (4.14) we extend $E_{1,1}$ (using the same abuse of notation) to:

$$E_{1,1}(\mathrm{s}^{-1}a;\mathrm{s}^{-1}b_1\otimes\ldots\otimes\mathrm{s}^{-1}b_n):=(-1)^{\gamma_2+|a|+1}\,\mathrm{s}^{-1}(a^1\cdot b_1)\otimes\ldots\otimes\mathrm{s}^{-1}(a^n\cdot b_n),$$

with

$$\begin{aligned}\gamma_2=\gamma_2(a)=\kappa_n(a)+\sum_{u=2}^{n}(|a^u|-1)\left(\sum_{s=1}^{u-1}|b_s|-u+1\right)\\ +\sum_{s=1}^{n}(|a^s|-1)(2n-2s+1)+\sum_{s=1}^{n-1}(|a^s|+|b_s|)(n-s),\end{aligned}$$

where

$$\kappa_n(a):=\begin{cases}\sum_{1\le 2s+1\le n}|a^{2s+1}| & \text{if } n \text{ is even;}\\ \sum_{1\le 2s\le n}|a^{2s}| & \text{if } n \text{ is odd.}\end{cases}$$

Finally, we find the operation in (2.3):

$$E_{1,1}(\mathrm{s}^{-1} a_1 \otimes ... \otimes \mathrm{s}^{-1} a_m; \mathrm{s}^{-1} b_1 \otimes ... \otimes \mathrm{s}^{-1} b_n) \\ := \sum_{l=1}^{m} (-1)^{\gamma_3} \mathrm{s}^{-1} a_1 \otimes ... (a_l^1 \cdot b_1) \otimes ... \otimes \mathrm{s}^{-1}(a_l^n \cdot b_n) \otimes ... \otimes \mathrm{s}^{-1} a_m$$

with

$$\gamma_3 = \sum_{u=1}^{n} |b_u| \left(\sum_{s=l+u}^{m} |a_s| + m - l - u + 1 \right) + \kappa_n(a_l) + \sum_{u=2}^{n} (|a_l^u| - 1) \left(\sum_{s=1}^{u-1} |b_s| - u + 1 \right) \\ + \sum_{s=1}^{n} (|a_l^s| - 1)(2n - 2s + 1) + \sum_{s=1}^{n-1} (|a_l^s| + |b_s|)(n - s),$$

where

$$\kappa_n(a) := \begin{cases} \sum_{1 \le 2s+1 \le n} |a^{2s+1}| & \text{if } n \text{ is even;} \\ \sum_{1 \le 2s \le n} |a^{2s}| & \text{if } n \text{ is odd.} \end{cases}$$

Definition 4.17. An ∞-morphism between two homotopy G-algebras, say Λ and Λ', is a morphism of unital DG-algebras between the associated bar constructions:

$$f : \mathcal{B}\Lambda \to \mathcal{B}\Lambda'.$$

Such a morphism is a collection of maps

$$f_n : \Lambda^{\otimes n} \to \Lambda', \quad n \ge 1,$$

of degree $1 - n$, satisfying the following relations (4.18) and (4.19),

$$\sum_{\substack{1 \le r \le k+l,\ 0 \le k_i \le 1 \\ k_1 + ... + k_r = k \\ l_1 + ... + l_r = l}} \pm f_r(E^{\Lambda}_{k_1,l_1} \otimes ... \otimes E^{\Lambda}_{k_r,l_r}) = \sum_{\substack{1 \le w \le l,\ 0 \le v \le 1 \\ i_1 + ... + i_v = k \\ j_1 + ... + j_w = l}} \pm E^{\Lambda'}_{v,w}(f_{i_1} \otimes ... \otimes f_{i_v}; f_{j_1} \otimes ... \otimes f_{j_w}), \tag{4.18}$$

for all $k \ge 1$, $l \ge 1$, and

$$\partial f_n = \sum_{j+k+l=n} \pm f_{n-1}(1^{\otimes j} \otimes \mu^{\Lambda} \otimes l^{\otimes l}) + \sum_{j+k=n} \pm \mu^{\Lambda'}(f_j \otimes f_k), \tag{4.19}$$

for all $n \ge 1$, where, ∂ is the differential of $\mathrm{Hom}(\Lambda^{\otimes n}, \Lambda')$.

Now we show that the homology of a homotopy G-algebra is a Gerstenhaber algebra. To fix the convention:

Definition 4.18. A Gerstenhaber algebra $(G, \cdot, \{\ ,\ \})$ graded commutative algebra $(G, \cdot)$ endowed with a degree 1 bracket,

$$\{\ ;\ \} : G \otimes G \to G$$

satisfying the following relations:

$$\{a, b\} = -(-1)^{(|a|+1)(|b|+1)} \{b, a\}; \tag{4.20}$$

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(|a|+1)|b|} b \cdot \{a, c\}; \tag{4.21}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\}. \tag{4.22}$$

Proposition 4.19. *Let $(\Lambda, d_\Lambda, \cdot, E_{1,k})$ be a homotopy G-algebra. Then the degree* 1 *bracket*

$$\{a;b\} = E_{1,1}(a;b) - (-1)^{(|a|-1)(|b|-1)}E_{1,1}(b;a)$$

defines a Gerstenhaber algebra structure on the homology $H(\Lambda, d_\Lambda)$.

Proof. The equality (4.15) shows the commutativity of the product of Λ up to homotopy. Indeed, it suffices to set $E^{\#}_{1,1}(a;b) := (-1)^{|a|}E_{1,1}(a;b)$ for each homogeneous element to obtain the desired homotopy. The symmetry condition (4.20) is satisfied by construction. The Jacobi relation (4.22) comes from (4.13). Indeed, let us first observe that applying (4.13) we have:

$$\begin{aligned} E_{1,1}(a;\{b;c\}) &= E_{1,1}(a;E_{1,1}(b;c)) - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(a;E_{1,1}(c;b)) \\ &= E_{1,1}(E_{1,1}(a;b);c) - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(E_{1,1}(a;c);b) + R \end{aligned}$$

where

$$\begin{aligned} R &= -E_{1,2}(a;b,c) - (-1)^{(|b|-1)(|c|-1)}E_{1,2}(a;c,b) \\ &\quad - (-1)^{(|b|-1)(|c|-1)}\Big(-E_{1,2}(a;c,b) - (-1)^{(|b|-1)(|c|-1)}E_{1,2}(a;b,c)\Big) \\ &= 0. \end{aligned}$$

From this

$$\begin{aligned} \{a;\{b;c\}\} =& E_{1,1}(a;\{b;c\}) - (-1)^{(|a|-1)(|b|+|c|)}E_{1,1}(\{b;c\};a) \\ =& E_{1,1}(a;\{b;c\}) \\ & - (-1)^{(|a|-1)(|b|+|c|)}\Big(E_{1,1}(E_{1,1}(b;c);a) - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(E_{1,1}(c;b);a)\Big) \\ =& E_{1,1}(E_{1,1}(a;b);c) - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(E_{1,1}(a;c);b) \\ & - (-1)^{(|a|-1)(|b|+|c|)}\Big(E_{1,1}(b;E_{1,1}(c;a)) - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(c;E_{1,1}(b;a))\Big) \\ & + L, \end{aligned}$$

where

$$\begin{aligned} L = -(-1)^{(|a|-1)(|b|+|c|)}\Big[& E_{1,2}(b;c,a) + (-1)^{(|c|-1)(|a|-1)}E_{1,2}(b;a,c) \\ & - (-1)^{(|b|-1)(|c|-1)}\Big(E_{1,2}(c;b,a) + (-1)^{(|b|-1)(|a|-1)}E_{1,2}(c;a,b)\Big)\Big]. \end{aligned}$$

By definition

$$\begin{aligned} -\{\{a;b\};c\} =& - E_{1,1}(E_{1,1}(a;b);c) + (-1)^{(|a|-1)(|b|-1)}E_{1,1}(E_{1,1}(b;a);c) \\ & + (-1)^{(|c|-1)(|a|+|b|)}\Big(E_{1,1}(c;E_{1,1}(a;b)) - (-1)^{(|a|-1)(|b|-1)}E_{1,1}(c;E_{1,1}(b;a))\Big). \end{aligned}$$

Thus we obtain,

$$\begin{aligned} \{a;\{b;c\}\} - \{\{a;b\};c\} =& - (-1)^{(|b|-1)(|c|-1)}E_{1,1}(E_{1,1}(a;c);b) \\ & - (-1)^{(|a|-1)(|b|+|c|)}E_{1,1}(b;E_{1,1}(c;a)) \\ & + (-1)^{(|a|-1)(|b|-1)}E_{1,1}(E_{1,1}(b;a);c) \\ & + (-1)^{(|c|-1)(|a|+|b|)}E_{1,1}(c;E_{1,1}(a;b)) \\ & + L. \end{aligned}$$

Finally, applying once again the equality (4.13) to both the 3-th and 4-th term, we obtain

$$\begin{aligned}
\{a;\{b;c\}\}-\{\{a;b\};c\} =& -(-1)^{(|b|-1)(|c|-1)}E_{1,1}(E_{1,1}(a;c);b)\\
&-(-1)^{(|a|-1)(|b|+|c|)}E_{1,1}(b;E_{1,1}(c;a))\\
&+(-1)^{(|a|-1)(|b|-1)}E_{1,1}(b;E_{1,1}(a;c))\\
&+(-1)^{(|c|-1)(|a|+|b|)}E_{1,1}(E_{1,1}(c;a);b)\\
&+L+L'\\
=& -(-1)^{(|b|-1)(|c|-1)}E_{1,1}(\{a;c\};b)+(-1)^{(|a|-1)(|b|-1)}E_{1,1}(b;\{a;c\})\\
&+L+L'\\
=&(-1)^{(|a|-1)(|b|-1)}\{b;\{a;c\}\}\\
&+L+L',
\end{aligned}$$

where

$$\begin{aligned}
L' = (-1)^{(|a|-1)(|b|-1)}\Big(E_{1,2}(b;a,c)+(-1)^{(|a|-1)(|c|-1)}E_{1,2}(b;c,a)\Big)&\\
+(-1)^{(|c|-1)(|a|+|b|)}\Big(-E_{1,2}(c;a,b)-(-1)^{(|a|-1)(|b|-1)}E_{1,2}(c;b,a)\Big)&.
\end{aligned}$$

The equality $L+L'=0$ is easily verified.

The Poisson relation (4.21) follows from the equations (4.16) and (4.17). Indeed, take $(-1)^{|b_1|}E_{1,2}(a;b_1,b_2)$ instead of $E_{1,2}(a;b_1,b_2)$ in (4.17), we obtain:

$$\begin{aligned}
\{a;bc\} =& E_{1,1}(a;bc)-(-1)^{(|a|-1)(|b|+|c|-1)}E_{1,1}(bc;a)\\
\sim& E_{1,1}(a;b)c+(-1)^{(|a|-1)|b|}bE_{1,1}(a;c)\\
&-(-1)^{(|a|-1)(|b|+|c|-1)}\Big(bE_{1,1}(c;a)+(-1)^{|c|(|a|-1)}E_{1,1}(b;a)c\Big)\\
\sim&\Big(E_{1,1}(a;b)-(-1)^{(|a|-1)(|b|+|c|-1)+|c|(|a|-1)}E_{1,1}(b;a)\Big)c\\
&+b\Big((-1)^{(|a|-1)|b|}E_{1,1}(a;c)-(-1)^{(|a|-1)(|b|+|c|-1)}E_{1,1}(c;a)\Big)\\
\sim&\{a;b\}c+(-1)^{(|a|-1)|b|}b\{a;c\},
\end{aligned}$$

where $\sim$ is the equivalence relation: $a\sim b$ iff a is homotopic to b.

The commutativity between the bracket and the differential follows from (4.15). Indeed,

$$\begin{aligned}
d_\Lambda\{a;b\} =& d_\Lambda E_{1,1}(a;b) - (-1)^{(|a|-1)(|b|-1)} d_\Lambda E_{1,1}(b;a) \\
=& - E_{1,1}(d_\Lambda a;b) - (-1)^{|a|}E_{1,1}(a;d_\Lambda b) + (-1)^{|a|}\Big(a\cdot b - (-1)^{|a||b|} b\cdot a\Big) \\
& -(-1)^{(|a|-1)(|b|-1)}\Big(E_{1,1}(d_\Lambda b;a) - (-1)^{|b|}E_{1,1}(b;d_\Lambda a) + (-1)^{|b|}\Big(b\cdot a - (-1)^{|a||b|}a\cdot b\Big)\Big) \\
=& - E_{1,1}(d_\Lambda a;b) - (-1)^{|a|}E_{1,1}(a;d_\Lambda b) \\
& - (-1)^{(|a|-1)(|b|-1)}\Big(E_{1,1}(d_\Lambda b;a) - (-1)^{|b|}E_{1,1}(b;d_\Lambda a)\Big) \\
& + (-1)^{|a|}\Big(a\cdot b - (-1)^{|a||b|}b\cdot a\Big) - (-1)^{(|a|-1)(|b|-1)+|b|}\Big(b\cdot a - (-1)^{|a||b|}a\cdot b\Big) \\
=& - E_{1,1}(d_\Lambda a;b) - (-1)^{|a|}E_{1,1}(a;d_\Lambda b) \\
& - (-1)^{(|a|-1)(|b|-1)}\Big(E_{1,1}(d_\Lambda b;a) - (-1)^{|b|}E_{1,1}(b;d_\Lambda a)\Big) \\
=& - \{d_\Lambda a;b\} - (-1)^{|a|}\{a;d_\Lambda b\}.
\end{aligned}$$

■

Proposition 4.20. *An ∞-morphism of homotopy G-algebras induces in homology a morphism of Gerstenhaber algebras.*

Proof. Let us take the following convention $f_i := \widetilde{f}_i(\mathrm{s}^{\otimes i})$. Then the equation (4.18) reads for $k = l = 1$:

$$E_{1,1}(f_1(a);f_1(b)) - f_1E_{1,1}(a;b) = (-1)^{|a|-1}f_2(a;b) + (-1)^{|a|(|b|-1)}f_2(b;a). \tag{4.23}$$

We obtain

$$\begin{aligned}
\{f_1(a);f_1(b)\} &= E_{1,1}(f_1(a);f_1(b)) - (-1)^{|a|-1)(|b|-1)}E_{1,1}(f_1(b);f_1(a)) \\
&= f_1E_{1,1}(a;b) + (-1)^{|a|-1}f_2(a;b) - (-1)^{|a|(|b|-1)}f_2(b;a) \\
&\quad - (-1)^{|a|-1)(|b|-1)}\Big(f_1E_{1,1}(b;a) + (-1)^{|b|-1}f_2(b;a) - (-1)^{|b|(|a|-1)}f_2(a;b)\Big) \\
&= f_1(\{a;b\}).
\end{aligned}$$

■

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